FORMAL AND MATERIAL CONSEQUENCE, DISJUNCTIVE SYLLOGISM AND GAMMA

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The study of medieval logic provides an opportunity for interaction between contemporary philosophical reflections and the thought of our predecessors in the Middle Ages. Sometimes the result is that recent insights are able to inform and clarify debates long since lost in obscurity; at others, the freshness and distinctive perspective which the medievals had on a problem feeds through to encourage us to overcome a narrowness and myopia in our own thinking, helping us to advance contemporary thought by drawing on the labours of those who have gone before.

I hope that in time it will be possible for the study of the theory of consequence in the Middle Ages to aid in the resolution of a dispute over the right criterion of validity which has dominated contemporary philosophy of logic for decades. For now, however, I must concentrate on a smaller endeavour. For there have been advances in our understanding of certain logical concepts in the last thirty or so years, and with the help of these discoveries I aim to show how a certain strand of thought among the medievals about the notion of consequence was one of great importance, and to explain what they were driving at. But I also want to exhibit a logical framework for tackling a rather specific problem which I think can be of wider use. Within a spectrum of modern interpretations of implication, medieval discussions can be helpfully fitted.

1. The Paradoxes of Implication

First then, I want to point to some important recent reflections on the notion of validity, or consequence.

The focus is the principle

ex impossibile sequitur quodlibet (EIQ, for short)

and its natural correlate

necessarium sequitur a quolibet (NAQ)

The plausibility of these principles rests on their following immediately from a natural account of logical consequence, namely, that an argument is valid if and only if it is impossible for the premises to be true and the conclusion false. For if the premises cannot all be true, then they cannot all be true together with the falsity of the conclusion; and if the conclusion cannot be false, then it cannot be false jointly with the truth of the premises.

In this century, most of the opposition to these principles has rested on their intuitive implausibility, if necessary backed up by looking at particular instances. Hence they are referred to as "paradoxes of implication": does implication, or logical consequence, really have these paradoxical properties? Thus, for example, Nelson wrote: "These paradoxes seem so utterly devoid of rationality that I consider them a *reductio ad absurdum* of any view which involves them."[1]

In the last thirty years, the creation of the most fully worked out system of logic directly rival to classical logic in its rejection of these principles, namely, relevant (or relevance) logic, arose directly from a paper by Ackermann, in which he wrote: "daß das Vorhandensein einer Aussage, die von allen impliziert wird oder die alle anderen impliziert, nicht dem Begriff der Implikation als eines logischen Zusammenhangs zwischen zwei Aussagen gerecht wird."[2] He also wrote

> Da aber auch diese Formeln [B < ¬(A ∧ ¬A) und A ∧ ¬A < B] vom Gesichtspunkt des logischen Zusammenhangs zwischen Vorder- und Hinterglied der Implikation aus zweifelhaft erscheinen, habe ich ein System der strengen Implikation aufgestellt, in dem diese und ähnliche Formeln nicht abgeleitet werden können.[3]

Even C.I. Lewis, later to become the most famous defender in recent times of the principles, started his search for a logic in which to express entailment with an open mind as to their correctness:

> That the merely contrary to fact implies anything is repugnant to common sense. But does the impossible—the absurd supposition—imply anything and everything? And is the necessarily true, whose denial is absurd, implied by any proposition whatever?[4]

[1] E. Nelson, "On three logical principles in intension", *The Monist*, 43, 1933, 268–84; p. 271.

[2] W. Ackermann, "Begründung einer strengen Implikation", *Journal of Symbolic Logic*, 21, 1956, 113–28; p. 113.

[3] W. Ackermann, "Über die Beziehung zwischen strikter und strenger Implikation", *Dialectica*, 12, 1958, 213–21; p. 214. Here Ackermann uses "<" for implication, "∧" for conjunction and "¬" for negation.

He includes them in his paper by including the postulate S9. But he adds:

> If one object to the notion that absurdities imply anything and that the necessarily true is implied by anything, then [substituting M6 for S9] will eliminate the above theorems and others which have a like significance.[5]

But as is well known, Lewis endorsed these principles. He gave two different arguments for their acceptability. The first uses Antilogism, from the case of Simplification,

$$(p \,\&\, {\sim}r) \rightarrow p$$

to

$$(p \,\&\, {\sim}p) \rightarrow r.$$

He settled finally on what are often known as the Lewis arguments:[6]

1.
$$p \,\&\, {\sim}p$$
$$p$$
$$p \vee q$$
$${\sim}p$$
$$q$$

2.
$$p$$
$$(p \,\&\, q) \vee (p \,\&\, {\sim}q)$$
$$p \,\&\, (q \vee {\sim}q)$$
$$q \vee {\sim}q$$

Anderson and Belnap took Ackermann's work, to build a coherent system in which these straightforward derivations fail.[7] How was this possible? Take the second of Lewis' arguments for EIQ. It depends on four principles:

[4] C.I. Lewis, "The Calculus of Strict Implication", *Mind*, 23, 1914, 240–7; p. 245.

[5] l.c., p. 246.

[6] In this paper, I shall use logical symbols as follows:

Symbol	Name of symbol	Approx. meaning
~	tilde	not
&	ampersand	and
∨	vel	or
→	arrow	if . . . then, implies

"¬" will be used only in §7, with a special meaning.

[7] A. Anderson/N. Belnap, *Entailment*, vol. I, Princeton: Princeton U.P., 1975.

1) Simplification
2) Addition
3) Disjunctive Syllogism
4) Transitivity, or Cut.

Transitivity, in the form "whatever is antecedent to the antecedent is antecedent to the consequent" (sometimes known as prefixing) and "whatever is consequent to the consequent is consequent to the antecedent" (or "suffixing") was commonly accepted by the medievals. Its rejection has been explored by Tennant[8] and others. But Anderson and Belnap retain it, as did Tarski, as essential to the idea of a chain of deduction. Simplification and Addition often go together, linked by the De Morgan principles. They were rejected by Nelson and Parry.[9] Again, Anderson and Belnap retain them, as characteristic of the truth-functional nature of '&' and '$\vee$'. It is Disjunctive Syllogism which they reject, or at least, Disjunctive Syllogism as applied to the outcome of a case of Addition.

The principle of Disjunctive Syllogism, that *ex disiunctiva cum contradictoria unius partis ad reliquam partem est bona consequentia*, was accepted by most medieval writers on consequence.[10] How can

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[8] N. Tennant, *Anti-Realism and Logic*, Oxford: Clarendon Press, 1987, p. 190.

[9] E. Nelson, "Intensional Relations", *Mind*, 39, 1930, 440–53; E. Nelson, op. cit. (n. 1); W. Parry, "Ein Axiomensystem für eine neue Art von Implikation (Analytische Implikation)", *Ergebnisse eines mathematischen Kolloquiums*, 4, 1933, 5–6.

[10] See, for example, Ockham, *Summa Logicae* (Opera philosophica I), ed. P. Boehner, G. Gal and S. Brown, St. Bovaventure, N.Y.: The Franciscan Institute, 1974, p. 350; Pseudo-Scotus, *In Librum Primum et Secundum Priorum Analyticorum Aristotelis Quaestiones* (short title: *"In Priora"*), in J. Duns Scotus, *Opera Omnia* ed. L. Wadding, Lyon 1639; repr. Paris: Vivès, 1891, vol. 2, 81–197, p. 184, cited in L. Pozzi, *Le "Consequentiae" nella logica medievale* (short title: *"Le "Consequentiae""*), Padua: Liviana, 1978, p. 166; Strode, in W. Seaton, *An edition and translation of the "Tractatus de Consequentiis" of Ralph Strode, fourteenth century logician and friend of Geoffrey Chaucer* (short title: *"De consequentiis"*), Ph.D. dissertation, University of California, 1973, p. 31; Paul of Venice, *Logica Magna*, Venice, 1499, f. 133^{rb}.

I would like to comment here on the identity of Pseudo-Scotus. Pozzi (l.c.) cites extracts from the *Prior Analytics* commentary printed in the Wadding/Vivès edition, and gives John of Cornwall as the author. When, some years ago, I was puzzled about the possible identification of the author of both the Commentary on the *Prior Analytics* and that on the *Posterior Analytics* as John of Cornwall, I wrote to Father Osmund Lewry, and now draw some of the following remarks from his reply. There are two pairs of questions: first, are the two texts ascribed to Scotus authentic? Secondly (if not), who is the author of each? For the commentary on the *Posterior Analytics* the questions are dealt with together: no manuscript attributes the work to Duns Scotus himself, and the colophon in the copy in Oxford Magdalen 250

Anderson and Belnap justify their rejection of it? I think their point can usefully be understood in the following way. In Lewis' proof, we use Disjunctive Syllogism to move from '$\sim p$' to q. What in general do we need to know to be justified in inferring q from '$\sim p$'? Obviously, we need the premise that if '$\sim p$' is true then so is q, that is, that '$\sim p \rightarrow q$' is true. The medievals recognised this when they said that the truth-conditions of a conditional are the same as the criteria for a valid consequence. In general, we have what is often nowadays called the Deduction Equivalence, or Deduction Theorem:

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expressly says, "datae a Johanne de Sancto Germano de Cornubia", i.e., John of Cornwall. The case is harder with the commentary on the *Prior Analytics*. Its authenticity was questioned by B. Gölz ("Die echten und unechten Werke des Duns Skotus nach dem gegenwärtigen Stand der Forschung", in *Siebte Lektorenkonferenz der deutschen Franziskaner für Philosophie und Theologie*, Werl i.W.: Franziskus-Druckerei, 1934, 53–60, p. 56); but references to the proof that it is not by Scotus lead to an apparently unpublished investigation of Longpré's of around 1935–7. For example, Bochenski wrote ("Notes historiques sur les propositions modales", *Revue des sciences philosophiques et théologiques*, 26, 1937, p. 688, n.2): "Nous devons cette information au R.P.E. Longpré qui a bien voulu nous la communiquer dans une lettre de 17 Janvier 1937. Les raisons avancées par le savant historien du scotisme nous paraissent péremptoires; esperons que le R.P. les publiera bientot."

Osmund Lewry concluded his letter: "For the *Posteriora*, . . . Grabmann, apparently in an older edition of the *Lexicon für Theologie und Kirche*, put forward the name of Henry Bate of Malines as a Pseudo-Scotus in this connection . . . I am not aware of any attempt to connect the *Priora* with John of Cornwall." I have not been able to check this remark about Grabmann. U. Smeets (*Lineamenta bibliographiae Scotisticae*, Rome, 1942, pp. 15–16) and others are careful to make the attribution only of the *Posteriora* to John of Cornwall. However, the commentary on the *Priora* was linked with him by A.C.S. McDermott ("Notes on the Assertoric and Modal Propositional Logic of the Pseudo-Scotus", *Journal of the History of Philosophy*, 10, 1972, 273–306, pp. 273–4): "R.P. E Longpré's examination of the material in question [viz pp. 1–347 of vol. II of the Vivès edition, i.e. the *Priora* and *Posteriora* commentaries] led him to the now generally accepted conclusion (ca. 1936) that Wadding had erred in ascribing these pages to Duns Scotus; so another "Pseudo-Scotus" was born, this one perhaps John of Cornwall." She notes Bochenski's claim ("De consequentiis scholasticorum earumque origine", *Angelicum*, 15, 1938, 92–109, p. 105, n. 7) that there appears to be a doctrinal inconsistency between Books I and II of the *Priora* commentary, suggesting that they are either not by the same author, or at least were not written together. She rejects his argument, asserting that her study of the two works uncovers no incompatibility at all. But McDermott gives no argument for her tentative ascription of both works, the *Priora* and the *Posteriora* alike, to John of Cornwall.

Pozzi recounts none of this. Simply to add John of Cornwall's name in parentheses after the designation "Pseudo-Scotus" as author of the *Priora* appears to go beyond the present evidence. Unless research has recently been done of which I am unaware, there is a need here for further investigation.

B follows from A and other premises $A_1, \ldots, A_n$
iff
'A → B' follows from $A_1, \ldots, A_n$.

In particular,

q follows from p and '~p'
iff
'~p → q' follows from p.

Hence, if EIQ were valid, it would follow that '~p → q' followed from p alone.

Another case of the Deduction Equivalence is this:

q follows from '~p' and q
iff
'~p → q' follows from q.

Given the validity of Simplification, that—in this case—q follows from '~p' and q, it follows that '~p → q' follows from q alone. These two results immediately entail that the conditional is truth-functional. Clearly, if p is true and q is false, 'p → q' is false; while if p is false, or if q is true, it follows from the above considerations that 'p → q' is true.

Hence if one accepts the Deduction Equivalence, one can see that all the Lewis proof does is spell out the intimate connection between EIQ and Disjunctive Syllogism. They stand or fall together. For Anderson and Belnap, they fall. Later, I will show how these insights can be brought together into the construction of a formal system (or in fact, a series of formal systems) within which questions of the validity of schemata such as EIQ and NAQ can be usefully posed. But first I want to turn back to the Middle Ages, to consider how the medievals reacted to these puzzles.

2. Formal and Material Consequence

At the Symposium on Medieval Logic and Semantics at Poitiers in 1985, C. Martin[11] reminded us that what I have called Lewis'

[11] C. Martin, "Embarrassing arguments and surprising conclusions in the development of theories of the conditional in the twelfth century", in *Gilbert de Poitiers et son contemporains, Acts of the 7th European Symposium on Medieval Logic and Semantics*, ed. J. Jolivet et A. de Libera, Napoli: Bibliopolis, 1987, 377–401.

proof of EIQ was known to the medievals at least as early as the twelfth century. Alexander Neckam recounts it as one of the puzzling proofs known to the school of Petit Pont in Paris when he was a student there (in the 1170s). Since that meeting at Poitiers, Martin has published a paper[12] in which he suggests that Lewis' proof may indeed be "William's Machine", whereby William of Soissons gave a general method of inferring an arbitrary proposition from a postulated contradiction, something that Abelard, reflecting on the plausible criterion for a valid consequence which we considered in §1, had realised should be possible.

But EIQ and NAQ puzzled the medievals, and they were not universally accepted as valid. Abelard himself, for example, described such consequences as *inconvenientia*, and was led to a stricter criterion of valid consequence. A valid consequence, he says, is one in which *non solum antecedens absque consequenti non potest esse verum*, but also where the antecedent in itself requires the consequent: *ex se ipso exigit.*[13] Referring to Priscian and Boethius, he explains this as a requirement that the consequent should be understood in, or already contained in the sense of the antecedent: *in ipso antecedentis sensu consequens iam contineri* (ibid.).

The Kneales[14] comment on the use of similar language in Kilwardby. Kilwardby makes a distinction between, on the one hand, natural and essential consequence, and on the other, accidental consequence. *Talis autem est consequentia secundum quam dicimus necessarium sequi ad quodlibet.* In contrast, in natural or essential consequence, *consequens naturaliter intelligitur in suo antecedente.*[15]

Such phrases may be very worthy; but what does this talk of "understood in" amount to? Ashworth mentions Gaetanus' examination of Ralph Strode's requirement for a formally valid consequence, that the consequent be formally understood in the antecedent.[16] There are many things it might mean. In fact, for Strode it did not exclude the Lewis argument.[17] Prior[18] drew a

[12] C. Martin, "William's Machine", *Journal of Philosophy*, 83, 1986, 564–72.
[13] P. Abelard, *Dialectica*, ed. L. de Rijk, Assen: van Gorcum, 1970, p. 284.
[14] W. and M. Kneale, *The Development of Logic*, Oxford: Clarendon, 1962, p. 276.
[15] I. Thomas, "Maxims in Kilwardby" (short title: "Maxims"), *Dominican Studies*, 7, 1954, 129–46, p. 137.
[16] E.J. Ashworth, *Logic and Language in the Post-Medieval Period*, Dordrecht: Reidel, 1974, pp. 128 ff.; Gaetanus, *Declarationes in Consequentias Strodi*, Venice, 1517, f. 34^{vb}.
[17] W. Seaton, *De consequentiis*, p. 31.
[18] A. Prior, *Formal Logic*, Oxford: Clarendon, 1962, p. 196.

similar conclusion when he said that what the Lewis argument shows (if it is sound) is that the meaning of a contradiction contains that of every other proposition.

Strode, along with most other logicians, distinguished formal from material consequence.[19] They did not all make the distinction in the same way, or assign arguments (consequences) to each side in exact agreement. They did not all even use the names "formal" and "material". Indeed, Strode and some others distinguish within "formal consequence" so-called *consequentia de forma* from *de materia*. It is *consequentia formalis de forma* which corresponds to what others called simply "formal consequence". Nonetheless, the basic idea is clear. From this perspective, EIQ and NAQ are materially, but not formally, valid, for arguments of the same form have true premises and false conclusion. For example,

If a man is an ass then God exists

and

If a man is an animal then Socrates exists

have the same form, but only in the first can the premise not be true without the consequent.

The Pseudo-Scot[20] shows how the formally and materially valid fit together. A materially valid consequence is essentially enthymematic. It needs the addition of a necessary truth to obtain a valid form. Suppose A is impossible, so that any other proposition, say B, may be said to follow from it by EIQ. Then '$\sim$A' is necessary, and by Lewis' argument, A and '$\sim$A' formally entail B, for Lewis' argument applies to any pair of contradictories. Then '$\sim$A' may be suppressed, and the materially valid inference of B from A remains.

Very many authors subscribed to this general picture: Sherwood, Kilwardby, Burleigh, Ockham, the "Logica Oxoniensis", Buridan, Albert of Saxony, Pseudo-Scotus, Strode, Lavenham, Ferrybridge, Fland, Peter of Mantua, Paul of Venice, Paul of Pergula, John

[19] Seaton, *De consequentiis*, pp. 1–2.

[20] Pseudo-Scotus, *In Priora*, p. 105—see Pozzi, L., *Le "Consequentiae"*, p. 155. See also Buridan, *Iohannis Buridani Tractatus de consequentiis*, ed. H. Hubien, Louvain: Publications Universitaires, Paris: Vander-Oyez S.A., 1976, I 4, p. 23.

Major and so on.[21] But there were dissenting voices. To two of them I now turn.

3. Relevance

One of the dissenters has been brought to our attention by Green-Pedersen.[22] He is Nicolaus Drukken of Dacia, writing in Paris in the 1340s. Like Abelard and Kilwardby, he rejects EIQ and NAQ, on the grounds that there is no affinity between antecedent and consequent. Hence the standard criterion of valid consequence, that the opposite of the consequent cannot stand with the antecedent, is necessary, but not sufficient. What we need is that the total significate of the consequent be signified by the antecedent. By this account, Drukken hopes to spell out the other idea that the antecedent should, in some way, include or contain the consequent.

The metaphor can in fact be spelled out in two ways: semantically, as Drukken does, requiring a connection of meaning, of the significate of antecedent and consequent; or inferentially, in requiring that the antecedent really be used to obtain the consequent. Anderson and Belnap, in their discussion of relevance as a criterion for entailment, have at various times appealed to both ideas.[23]

The fault with meaning connection is that it can only hope, once again, to be a necessary and not sufficient condition. For there are

[21] William of Sherwood, *Treatise on Syncategorematic Words*, trad. N. Kretzmann, Minneapolis: University of Minnesota Press, 1968, p. 123; Kilwardby—see Thomas, "Maxims", p. 137; Burleigh—see Pozzi, *Le "Consequentiae"*, p. 171, Ockham, *Summa Logicae*, p. 589—see Pozzi, p. 146; "Logica Oxoniensis"—see L. de Rijk, "Logica Oxoniensis", *Medioevo*, 3, 1977, 121–64, pp. 122–3; Buridan—see Pozzi, *Le "Consequentiae"*, p. 181; Albert of Saxony—l.c., p. 209; Pseudo-Scotus, *In Priora*, p. 105—see Pozzi, *Le "Consequentiae"*, p. 154; Strode—l.c., p. 237; Lavenham—see P. Spade, "Five logical tracts by Richard Lavenham", in *Essays for C. Pegis*, ed. J. O'Donnell, Toronto: Pontifical Institute of Medieval Studies, 1974, 70–124, p. 99; Ferrybridge—see Pozzi, *Le "Consequentiae"*, p. 262; Fland—see P. Spade, "Robert Fland's *Consequentiae*: an edition", *Medieval Studies*, 38, 1976, 54–84, p. 57; Peter of Mantua—see Pozzi, *Le "Consequentiae"*, p. 283; Paul of Venice, *Logica Magna*, ff. 139^{vb}–140^{vb}; Paul of Pergula, *Dubia [in Consequentias Strodi]*, Venice, 1517, f. 47^{vb}–48^{va}; Major—see A. Broadie, *The Circle of John Mair*, Oxford: Clarendon, 1985, p. 214. Some may object to my including, e.g., Sherwood and Ockham. But I do not read great significance into the fact that the only examples of materially valid consequences which they give are EIQ and NAQ.

[22] N. Green-Pedersen, "Nicolaus Drukken de Dacia's Commentary on the Prior Analytics—with special regard to the theory of consequences", *Cahiers de l'Institut du Moyen Age Grec et Latin*, 37, 1981, 42–69.

[23] See G. Iseminger, "Is relevance necessary to validity?", *Mind*, 89, 1980, 196–213.

clearly many propositions which are connected in meaning but which are not consequences of one another. At least in the sphere of propositional logic, Belnap[24] made a plausible case for requiring that A and B should share a propositional variable in order for A to entail B. But whereas Belnap is willing to accept the notion of an enthymematic inference (whose suppressed premise may contain the variable needed for relevance), Drukken goes further. Any valid consequence is formal, says Drukken, and accordingly rejects the very distinction of formal from material consequence. But this still does not answer the central question, the search for a sufficient condition of validity. For what we seek, in looking for a stricter requirement of valid consequence than the impossibility of the antecedent's obtaining without the consequent, is a sufficient condition. We already, in that standard condition, have a necessary condition. The trouble is that it allows arguments such as EIQ and NAQ to slip through, and we therefore are puzzled to know quite what will suffice for valid consequence.

The other possibility Anderson and Belnap consider is derivational utility. Not only must the antecedent have been used to obtain the consequent; further, if it was, then that gives unimpeachable grounds to say the consequence is valid. But this idea is empty until we have some idea what the criterion of "being used" is. Ashworth[25] rightly castigates Melanchthon for simply requiring of a valid consequence that it not violate the precepts of dialectic. For surely we need a criterion of valid consequence in order to test those putative precepts for correctness. To tell us that in order to know what a valid consequence is we need to know which consequences are valid is, as Plato said, "extremely dark counsel" (*Theaetetus* 209e).

But we need not to be so pessimistic. For logical theory is no different from scientific theory in being but a conjectured set of generalisations proposed as an explanation of phenomena, and defeasible in the light of counter-evidence. Knowledge requires, besides evidence, that what is known be true. If it is true—and is properly grounded—it is knowledge. If it turns out to be false—however good the evidence—then it was not. Our approach to logical

[24] N. Belnap, "Entailment and Relevance", *Journal of Symbolic Logic*, 25, 1959, 144–6.

[25] E.J. Ashworth, *Logic and Language in the Post-Medieval Period*, p. 127.

theory, therefore, should be similar to the way a scientific theory is developed. We start small, and increase its bounds slowly, going for correctness at first, completeness only as an eventual aim.

It would be a mistake, therefore, to use the evidence of the paradoxes of implication simply to remove part of the superstructure of classical logic, retaining what has not yet been shown to be faulty. Instead we should slowly build our theory from below, conjecturing general principles and examining them carefully before placing too much confidence in them.

Consequence is a relation between antecedent and consequent. The immediacy with which the Deduction Equivalence extracts the paradoxes of implication from Simplification (as on p. 238 above) should give us pause. One approach, which I have urged elsewhere,[26] makes a radical distinction between intensional and extensional premise combination, and correspondingly between intensional and extensional connectives. Mark the extensional combination by the comma, as is usual, denoting set union; mark the intensional combination by a semi-colon. (Think of X;Y as the result of composing X with Y, as in functional application.[27]) We then obtain as introduction rules for '&' and '→':

$$\frac{X : A \qquad Y : B}{X,Y : A\&B}\ \&I \qquad\qquad (\&\text{-introduction})$$

$$\frac{X;A : B}{X : A \rightarrow B}\ \rightarrow I \qquad\qquad (\rightarrow\text{-introduction})$$

These schemata should be understood as follows: the colon separates the consequent of a consequence (or sequent) from its antecedent(s); in these displays, the sequent below the line (the conclusion of the rule) may be inferred from the sequent(s) above the line (the premises). A proof will consist of a tree (a partially ordered display) of applications of such rules, where the leaves of the tree (the initial sequents) will have the form A : A, for some formula A. We then say that X ⊢ A (A is derivable from X) if there is a proof whose final sequent is X : A.

[26] S. Read, *Relevant Logic*, Oxford: Blackwells, 1988; see also S. Read, *Proof Theory and Semantics for Relevant Logic*, Technical Report TR-ARP 13/87, Research School of Social Sciences, Australian National University, Canberra, 1987.

[27] In other words, X;Y entails C if X entails that Y entails C.

Let the introduction rule so stated (and similarly for other connectives) be taken to express the assertion conditions (the meaning) of the connective. Then Cut (or transitivity), as described in §1 above, constrains the elimination rule.[28] In general, whatever follows from the assertion conditions (say, proofs of A and of B) of a formula (in that case, 'A & B') follows from the formula. The elimination rule for '&', for example, is:

$$\frac{X : A \,\&\, B \qquad\qquad Y(A,B) : C}{Y(X) : C}\;\&E$$

where Y(A,B) shows an antecedent combination Y containing the extensional combination A,B as a part, and Y(X) results by substitution of X for that part.

Besides the operational rules (rules specific to different connectives), it is necessary to include here structural (or generic) rules, governing the identity of various combinations of comma and semi-colon. (They are often left implicit in presentations of classical logic, where the antecedent of a consequence is always a set.) Within this conception of paired introduction and elimination rules for the connectives, in particular, for '&', Simplification is a consequence of the principle of Strengthening the Antecedent (or Augmentation). Augmentation is captured by the imposition of the structural or generic rule:

$$\frac{X : A}{X,Y : A}\;EK$$

EK says that whatever entails A can be augmented extensionally and still entail A. The rule is also often called Weakening, Thinning or Dilution. "E" in "EK" signifies that the weakening holds for the extensional combination, comma; and the "K" alludes to Curry's combinator K, to which weakening corresponds.

EK, together with the &E rule, yields the inference commonly called Simplification, that A & B ⊢ A.

$$\frac{A \,\&\, B : A \,\&\, B \qquad\qquad \dfrac{A : A}{A,B : A}\,EK}{A \,\&\, B : A}\;\&E$$

[28] See M. Dummett, *Frege: Philosophy of Language*, London: Duckworth, 1973, p. 396.

Since the leaves both have the form of initial sequents, this constitutes a proof that A & B ⊢ A.

If we left it there, we would still succumb to the implicational paradoxes. For recall from §2 Pseudo-Scotus' explanation of the connection between formal and material consequence: a consequence is materially valid if and only if it can be converted to a formally valid consequence by the addition of a suppressed necessary premise.[29] Suppression, in the guise of Cut, gives us NAQ as a formal consequence immediately. Take any logical truth, e.g., A ∨ ~A:

$$\frac{\overset{\cdots}{: A \vee \sim A} \qquad\qquad \dfrac{A \vee \sim A : A \vee \sim A}{B, A \vee \sim A : A \vee \sim A}\,EK}{B : A \vee \sim A}\;Cut$$

Can we retain Cut and yet block this route to NAQ? One way, natural in the present context, is to distinguish an intensional identity element from the extensional identity element, ∅, the empty set. ∅ is an identity for comma (set union): X ∪ ∅ = ∅ ∪ X = X. What we need is an identity element, t, for the intensional combination, semi-colon. We start by letting t be a left identity for semi-colon: t; X = X. Then we say that ⊢ A holds if we can derive the sequent t: A, that is, if t ⊢ A. (∅: A drops out as irrelevant.) t and not ∅ then becomes "the logic", the mark of what is logically true.

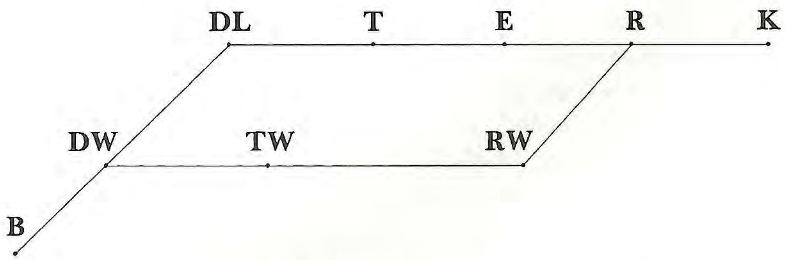
This blocks the paradoxical derivation above. For we need to add 't', denoting the logic itself, to our derivation of A ∨ ~A:

$$\frac{\overset{\cdots}{t : A \vee \sim A} \qquad\qquad \dfrac{A \vee \sim A : A \vee \sim A}{B, A \vee \sim A : A \vee \sim A}\,EK}{t,B : A \vee \sim A}\;Cut$$

But that is no surprise, for t,B: A ∨ ~A follows immediately from t: A ∨ ~A by EK. What we must ensure, by the theory of 't', is that the *extensional* addition of t here cannot be suppressed. The proof that "necessarium sequitur ad quodlibet" will accordingly fail. The derivation of NAQ is blocked.

[29] See also J. Bendiek, "Die Lehre von den Konsequenzen bei Pseudo-Scotus", *Franzikanische Studien*, 34, 1952, 205–34, p. 220.

Depending on quite how strong an introduction rule for negation we adopt, these ideas lead to a weak relevant logic, **B** or **DW** or **DL**.[30] The systems can be displayed as follows:



where systems above or to the right properly contain systems below and to the left. The logic is weak in its account of '$\rightarrow$': in **B** and **DW**, if '$A \rightarrow B$' is provable, and A and B contain only '$\rightarrow$' as connective, B is the same formula as A. It is a relevant logic in that it satisfies Belnap's variable-sharing condition mentioned earlier: if '$A \rightarrow B$' is provable then A and B share a variable.

B and **DW** have little to recommend them as the correct logic. Their virtue is that their logical theory consists of no more than is given by the meanings (assertion conditions) of the connectives '&', '$\vee$', '$\sim$' and so on, coupled to the Deduction Equivalence (consisting on the one hand of Conditional Proof, or $\rightarrow$ I, and on the other of Modus Ponendo Ponens, or $\rightarrow$ E). They provide a suitable base on which to impose further conditions on consequence—and to do so by concentrating on the theory of the semi-colon, the intensional premise combination. For example, we can require that semi-colon be associative, that it be commutative, that t be also a right identity (which of course, it will be if we opt for commutativity), and so on. Thereby we obtain stronger and stronger logics, **E**, **R** and even **K**, classical logic. What changes as we alter these structural rules governing semi-colon, is the theory of implication, whose meaning is tied to semi-colon by the $\rightarrow$ I rule. In **K**, the semi-colon takes on all aspects of comma, including Weakening (EK), and so the intensional/extensional distinction collapses and relevance disappears. Alone of the logics displayed, **K** lacks the variable-sharing

[30] See R. Routley et al., *Relevant Logics and their rivals*, Atascadero, Calif.: Ridgeview, 1982, pp. 287 sqq.; S. Read, *Proof Theory and Semantics for Relevant Logic*; and *Relevant Logic*, Ch. 4.

property. **R**, in which semi-colon is taken to be associative, commutative and to satisfy contraction (see below, p. 249), is the central relevant logic.

4. Aristotle's Thesis

Kilwardby's reason for rejecting EIQ and NAQ was theoretical. Including them in the logic led, he thought, to unintuitive consequences. So he rejected them not, or not only, because they seemed intuitively unacceptable in themselves, but because their inclusion came up against a heuristic falsifier.[31]

The problem stems from a notorious passage in Aristotle's *Prior Analytics*, 57a29 sqq., where Aristotle appears to enunciate as a general principle that no proposition follows from its contradictory. The context is his desire to point out that a valid syllogism may have false premises and a true conclusion. He sets out to show two things: first, that the falsity of the premises does not imply the falsity of the conclusion—that would be the fallacy of affirming the consequent. But equally, the falsity of the premises does not imply the truth of the conclusion. Since the syllogism is valid, the truth of the premises (say, A) entails the truth of the conclusion (B). So if we can show that

$$(A \rightarrow B) \rightarrow \sim(\sim A \rightarrow B)$$

it will follow that the falsity of the premises ($\sim A$) does not entail the truth of the conclusion, as required. Aristotle's argument is this: given $A \rightarrow B$, we obtain $\sim B \rightarrow \sim A$ by rule contraposition. Suppose $\sim A \rightarrow B$. Then $\sim B \rightarrow B$ by rule suffixing. (We'll discuss these inferences, and in particular, the distinction between, for example, contraposition and rule contraposition, further in §5.) But that is impossible, he says. So $\sim A \rightarrow B$ cannot be true, and so

$$(A \rightarrow B) \rightarrow \sim(\sim A \rightarrow B),$$

as required.

Many commentators have questioned whether Aristotle really meant here to deny that any proposition follows from its contra-

[31] Cf. I. Lakatos, "A renaissance of empiricism in the recent philosophy of mathematics?", in *Mathematics, Science and Epistemology* (Philosophical Papers, vol. 2), ed. J. Worrall and G. Currie, Cambridge: Cambridge U.P., 1978, 24–42, p. 36.

dictory. Geach, for example, believes that Aristotle meant to deny it only for A, E, I and O propositions, for the context shows he is concerned only with syllogistic propositions.[32] Pseudo-Scotus made much the same point: "Sic patet quod illa regula scilicet *ad idem esse et non esse non sequitur idem* intelligitur solum in simplicibus categoricis et de consequentia formali".[33] For Pseudo-Scotus accepted, we noted in §2, that NAQ holds as a material consequence. So in particular, a necessary proposition follows materially from its own contradictory. But not formally. So that even a necessarily true simple categorical such as

Deus est

is not formally entailed by its contradictory, nor by a proposition such as *tu sedes* and its contradictory, *tu non sedes*.

Kilwardby said the same. Aristotle's concern was natural and essential consequence. EIQ and NAQ hold only of accidental consequence, and *de tali* [scilicet *consequentia accidentalis*] *non intelligendus est sermo Aristotelis*.[34]

Not everyone has interpreted Aristotle in this way, however. Storrs McCall,[35] among others in recent times, has tried to construct a connexive logic in which Aristotle's thesis is true. It is useful here to distinguish the following theses:

$\sim(A \rightarrow \sim A)$	Aristotle's Thesis (equivalently[36], $\sim(\sim A \rightarrow A)$)
$(A \rightarrow B) \rightarrow \sim(A \rightarrow \sim B)$	Boethius' Thesis
$\sim((A \rightarrow B) \,\&\, (A \rightarrow \sim B))$	Strawson's Thesis

None of these can be added consistently to classical logic, or even to Anderson and Belnap's preferred logics **R** and **E**. The reason is that those logics contain Modus Tollendo Tollens, or ~I ('not'-introduction), in the form:

$$\frac{X;A : B \qquad Y : \sim B}{X;Y : \sim A}\ \sim I$$

[32] P. Geach, "Aristotle on conjunctive propositions", in *Logic Matters*, Oxford: Blackwells, 1972, 13–27, p. 25.

[33] Pseudo-Scotus, *In Priora*, pp. 185–6—Pozzi, *Le "Consequentiae"*, p. 169.

[34] Thomas, "Maxims", p. 137.

[35] S. McCall, "Connexive Implication", *Journal of Symbolic Logic*, 31, 1966, 415–33.

[36] In the context of Double Negation, that $A \dashv\vdash \sim\sim A$.

together with Simplification (see above, p. 245), that A follows from A & B. The following proof then contradicts Aristotle's Thesis:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Simplification}}{A \,\&\, \sim A : A}}{t;A \,\&\, \sim A : A}\ t= \qquad \dfrac{\text{Simplification}}{A \,\&\, \sim A : \sim A}}{t;A \,\&\, \sim A : \sim(A \,\&\, \sim A)}\ \sim I.}{t : (A \,\&\, \sim A) \rightarrow \sim(A \,\&\, \sim A)}\ \rightarrow I$$

Hence $\vdash (A \,\&\, \sim A) \rightarrow \sim(A \,\&\, \sim A)$. Thus $A \,\&\, \sim A$ is a formula which entails its own negation, contrary to Aristotle's Thesis, that no formula does. Since Boethius' Thesis entails Aristotle's Thesis, it follows that, Aristotle's Thesis being false, so is Boethius'.

It might be thought that Boethius' and Strawson's Theses are equivalent. But that presupposes that

$$A \rightarrow B \dashv\vdash \sim(A \,\&\, \sim B)$$

and this equivalence fails even in **R** and **E**. Clearly, by Simplification, Antilogism (that if $X;A \vdash B$, then $X; \sim B \vdash \sim A$) and t =, we can prove that $\sim A \vdash \sim(A \,\&\, \sim B)$, and so, if the above equivalences held, we would obtain, by Cut,

$$\sim A \vdash A \rightarrow B.$$

But **R** and **E** were designed specifically to exclude such a truth-functional and paradoxical account of '→'. In **R** and **E**, Boethius' Thesis entails Strawson's but not vice versa. In **DW** even that entailment fails, for **DW** lacks both Permutation and Contraction[37] as theses about '→'.

Simplification and ~I hold in **DW** and all systems containing it. A connexive logic which will accommodate Aristotle's and Boethius' Theses must therefore part ways with classical and relevant logics in a deep way. McCall's logic is Post-complete, as is classical logic; that is, adding any non-thesis as an axiom would entail inconsistency.[38]

[37] Permutation says that $A \rightarrow (B \rightarrow C) \vdash B \rightarrow (A \rightarrow C)$. Contraction says that $A \rightarrow (A \rightarrow B) \vdash A \rightarrow B$.

[38] A. Church, *An Introduction to Mathematical Logic*, Princeton U.P., 1956, §18.

(**R**, **E**, **DW** and so on, being proper subsystems of classical logic, are far from Post-complete.)

5. Rules and Rules

I have referred frequently to such notions as *rule* contraposition, *rule* Simplification, and so on. What exactly does this mean? Whether an inference holds in thesis form or in rule form is an important distinction,[39] and it will take us back to the medievals and EIQ and NAQ. Here are some examples from classical logic which should be familiar.

The rule of substitution in classical logic says that if A is a theorem containing the propositional variable p at one or more occurrences, and A′ results from A by replacing every occurrence of p by the formula B, then A′ is also a theorem. It is important to realise that Substitution holds only in rule form, that is,

$$\vdash A \Rightarrow \vdash A'.$$

In thesis form it is false, that is, it is not true that

$$\vdash A \rightarrow A'.$$

For if so, we could take ‘p’ for A and ‘q’ for B, and infer

$$\vdash p \rightarrow q,$$

which is clearly invalid, and so from which, since classical logic is Post-complete, we would obtain triviality.

Another example is the rule of Necessitation in normal modal logic. It says that if A is a theorem, then so too is □A (where ‘□’ is the sign for necessity). Again, it is important to realise that Necessitation holds only in rule form, that is,

$$\vdash A \Rightarrow \vdash \Box A$$

and not in thesis form, that is, it is not true that

$$\vdash A \rightarrow \Box A.$$[40]

[39] See, for example, Routley et al., *Relevant Logics and their rivals*, p. 45.

[40] See, e.g., G. Hughes and M. Cresswell, *A Companion to Modal Logic*, London/New York: Methuen, 1984, Ch. 1. By the Deduction Equivalence, this last sequent is equivalent to A ⊢ □A, which must fail. In other words, □A does not follow from the *assumption* of A.

For if it were, modalities would collapse, and A and □A would be equivalent.

In **B** and **DW**, a number of interesting rules hold only in rule form, and not in thesis form. We noted in §1 two rules often stated by the medievals, namely, prefixing and suffixing.[41] They may hold in a logic either in rule form, or in thesis form (then often known nowadays as *B* and *B*′, after the corresponding combinators):

$$\vdash A \rightarrow B \Rightarrow \vdash (C \rightarrow A) \rightarrow (C \rightarrow B) \text{ (rule prefixing)}$$
$$\vdash A \rightarrow B \Rightarrow \vdash (B \rightarrow C) \rightarrow (A \rightarrow C) \text{ (rule suffixing).}$$

In **R** (indeed, in **TW**[42]) they hold in thesis form:

$$\vdash (A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B)) \qquad (B)$$
$$\vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)) \qquad (B').$$

Almost every treatise on consequences lists these principles:

Quicquid sequitur ad consequens, sequitur ad antecedens.
Quicquid antecedit ad antecedens, antecedit ad consequens.

What distinguishes **DW** from **B** is that contraposition holds in thesis form in **DW**, but only in rule form in **B**:

$$\vdash A \rightarrow B \Rightarrow \vdash {\sim}B \rightarrow {\sim}A \qquad \text{(rule contraposition).}$$

It is difficult to tell whether the medievals made this distinction between rules and theses, or whether what they say of the rules can help one decide if they would accept the stronger, thesis, form in any particular case. However, there is one place at which the distinction is crucial to understanding the text, and it takes us back full-circle to Disjunctive Syllogism, EIQ and William's machine.

6. Gamma

I said at the end of §2 that I would pick out two dissenting voices on the question of the validity of EIQ and NAQ. The second is Domingo de Soto, writing in Paris in the early sixteenth century. De Soto also realised that EIQ and NAQ were incompatible with

[41] See, for example, Pozzi, *Le "Consequentiae"*, p. 70.

[42] See R. Brady, "Natural Deduction Systems for some quantified relevant logics", *Logique et Analyse*, 27, 1984, 355–77, p. 359.

Aristotle's Thesis. The example he considered had been discussed by many authors before him:[43]

si Deus non est, Deus est.

Since the antecedent is impossible, and the consequent necessary, the consequence is valid, by EIQ and NAQ. But it directly contradicts Aristotle's Thesis. Aristotle did not say he meant his point only for the case of syllogistic propositions, and anyway, says de Soto, there is clearly no relation between such propositions as "you are a stone" and "you are God", and so neither follows from the other, even though both are impossible.

> Nulla est penitus habitudo inter hoc quod est te esse lapidem et te esse deum, ergo neutrum sequitur ad alterum, nihil enim dicitur sequi ad alterum nisi includatur aut quodammodo proportionabiliter se habeat ad illud.[44]

Accordingly, simply to say that a consequence is valid if it cannot be as is signified by the antecedent without it also being as is signified by the consequent, cannot be the right account of valid consequence.

But a modified attack on Aristotle's Thesis threatens. Even if *si Deus non est, Deus est* is rejected as an invalid consequence, at least it would seem that from any pair of contradictories anything whatever follows (including, therefore, its own contradictory). The argument is the familiar machine of William's (and Lewis'):

> Sequitur bene. Petrus disputat et petrus non disputat, ergo petrus non disputat, et ex alia parte, petrus disputat et petrus non disputat, ergo petrus disputat, quia utrobique arguitur a copulative ad eius alteram partem, rursus, petrus disputat, ergo petrus disputat vel homo est lapis, a parte disiunctive ad totam rursus petrus disputat vel homo est lapis et petrus non disputat, que erat consequens prime consequentie, ergo homo est lapis, a tota disiunctiva cum destructione unius partis ad positionem alterius, ergo de primo ad ultimum, petrus disputat et petrus non disputat, ergo homo est lapis, quia quicquid sequitur ad consequens bone consequentie sesivitur (*sic*) ad eius antecedens, et eadem arte ex eodem antecedenti poteris inferre quodcumque consequens.[45]

[43] See, for example, Ralph Strode, *Consequentie cum commento Alexandri Sermoneta*, Venice, 1517, f. 12^{v} ff.; cf. Pseudo-Scotus, *In Priora*, p. 183—Pozzi, *Le "Consequentiae"*, p. 165.

[44] Domingo de Soto, *Introductiones Dialecticae*, Burgos, 1529, f. 74^{rb}.

[45] l.c., f. 74^{va}.

De Soto's reply is to claim that a pair of contradictories can be understood in two different ways: in one way, they may be taken absolutely, and not merely on assumption, and then one can conclude that the one utterly rules out the other. But in another way, they may be taken merely as conceded for the sake of argument, and then neither utterly rules out the other, for they have both, for the sake of argument, been conceded to be true. Consider then, the step of Disjunctive Syllogism in the argument:

petrus disputat vel homo est lapis, et petrus non disputat
ergo homo est lapis.

If those contradictories were taken absolutely, the inference would be good. But if they are conceded simply for the sake of argument it is not. In that case, we do not really have a case of the commonly accepted mode of reasoning (see above, p. 236),

ex tota disiunctiva cum destructione unius partis sequitur reliqua pars.

I think we should interpret de Soto here to be admitting that Disjunctive Syllogism holds in rule form but not in thesis form. That is, de Soto concedes that

$$\vdash A \vee B, \vdash {\sim}A \Rightarrow \vdash B$$

but he denies that

$$\vdash (A \vee B) \,\&\, {\sim}A \rightarrow B.$$

Is this interpretation plausible? It is plausible for two reasons. First, it clearly fits the text. Secondly, it has been shown that although Disjunctive Syllogism fails in thesis form in the relevant logics **E** and **R** (a simple matrix shows it, and as we saw, it is precisely by rejecting Disjunctive Syllogism in thesis form that these logics block the Lewis argument), the rule form of Disjunctive Syllogism does hold in them. It has come to be known as γ (Gamma):

$$\vdash A \vee B, \vdash {\sim}A \Rightarrow \vdash B \text{ (Gamma).}$$

Gamma holds of a theory T if whenever $A \vee B \in T$ and ${\sim}A \in T$, $B \in T$. **R** and **E** are Gamma-theories.[46]

[46] See, for example, J.M. Dunn, "Relevance Logic and Entailment", in *Handbook of Philosophical Logic*, vol. III: Alternatives to Classical Logic, ed. F. Guenthner and D. Gabbay, Dordrecht: Reidel, 1986, 154–224, pp. 150–65. The name Gamma derives from the fact that Anderson and Belnap obtained their initially

De Soto's *Summulae* went through three editions, and the passage cited above appears only in the first edition. It was successively revised for the second and third editions. In the second edition, de Soto omits reference to the distinction between taking a proposition absolutely or only for the sake of argument. Instead, he rejects William's machine in a different way—revising a passage which also occurred in the first edition. The revisions have been set out and de Soto's analysis discussed by d'Ors.[47] D'Ors concludes that "si la crítica de Soto es correcta, nuestra lógica proposicional debe ser revisada", adding in a footnote, "Obsérvese que la crítica de Soto impone un claro límite al «principio de sustutividad de equivalentes». Si en «$(A \wedge \neg A) \rightarrow A$», que es una fórmula válida y demostrable, se sustituye «$(A \wedge \neg A)$» por su equivalente «$(B \wedge \neg B)$» se obtiene la fórmula «$(B \wedge \neg B) \rightarrow A$», que no sería ya, según la opinión de Soto, demostrable."[48]

But this is incorrect. Certainly revision is needed. But it need not affect the substitutivity of equivalents. Given de Soto's claim that a contradiction does not entail an arbitrary proposition, there is no reason to accept that any pair of contradictiories are equivalent. Indeed, in **R** and **E** (and the weaker logics) they are not. In those logics, substitutivity of equivalents is valid, as is Simplification, while EIQ and Disjunctive Syllogism fail. In this respect, the logics are fully extensional. (They are intensional in that they admit no finite characteristic matrix.)

The idea that Disjunctive Syllogism might fail in thesis form but still hold in rule form was not original to de Soto. The Cologne commentators had voiced the same thought at the end of the fifteenth century. It is false, they said, that anything follows from an impossible proposition. Only what is included in it follows, and not everything is so included: *quicquid sequitur ad alterum necessario includitur in*

preferred logic **E** from Ackermann's system Π' by (inter alia) dropping from his axiomatic formulation the third of his four rules of inference, (α), (β), (γ), (δ). Perhaps the most important recent discovery in the theory of relevant logic is that Gamma fails for $R^{\#}$, namely, the theory obtained by adding the five Peano axioms to a first order logic erected on **R**. If Gamma held for $R^{\#}$, then classical Peano arithmetic would be a conservative extension of its positive fragment. Harvey Friedman has shown that it is not.

[47] A. d'Ors, "Las Summulae de Domingo de Soto" (short title: "Las Summulae"), *Anuario Filosofico*, 16/1, 1983, 209–217, pp. 213–7.

[48] l.c., p. 217. Note that d'Ors here uses "$\neg$" for negation, "$\wedge$" for conjunction and "$\rightarrow$" for implication.

eo, sed quodlibet non includitur in uno impossibili licet multo includantur in eo.[49] But as with de Soto, they must refute not only the use of the standard account of consequence (that it be impossible for the antecedent to be true without the consequent), but also William's machine. They too focus on the final inference, the use of Disjunctive Syllogism:

> Dicendum quod ultima consequentia (qua arguitur in disiunctivis) non valet quia ille propositiones sortes currit et sortes non currit capiuntur dupliciter, uno modo absolute et secundum se et sic contradicunt sibi mutuo. Alio modo accipiuntur ut concesse ab aliquo simul gratia disputationis et sic non contradicunt quia simul admittuntur ab aliquo tanquam vere. ut scilicet posset cognoscere an quodlibet inconveniens posset sequi ad contradictoria et sic non destruunt se et per consequens in ultima consequentia non arguitur a destructione unius partis disiunctive quia ille propositiones sunt prius concesse sed quod conceditur non destruitur.[50]

7. Boolean Negation

A puzzle remains, however. What did de Soto and the Cologne commentators mean when they said that when *sortes currit* and *sortes non currit* are taken for the sake of argument, they do not contradict one another, and we do not really have a case of the familiar mode of reasoning of Disjunctive Syllogism? De Soto wrote that when two contradictories are taken *gratia disputationis*,

> tunc certe neutra destruit alteram, quia ex hoc quod una conceditur vera, non sequitur alteram esse falsam, cum ambe conceduntur vere. . . si accipiantur ut concesse in primo antecedenti, certe nihil valet, quia ex hoc quod negativa est vera non sequitur data suppositione quod affirmativa sit falsa, cum causa disputationis ambe sint concesse, quare non arguitur a tota disiunctiva cum destructione unius partis. Nam destruere unam propositionem non est ponere eius contradictoriam, sed tollere eius veritatem, quod tali casu non fit per positionem contradictorie.[51]

The distinction our medieval authors are making will become clear if we dig a bit deeper into contemporary relevant logic.

By 1970 the formal theory of relevant logic was well established.

[49] ⟨anon.⟩ *Copulata super omnes tractatus parvorum logicalium Petri Hispani. . .*, Cologne 1493, f. 103^{v}.

[50] l.c., f. 104^{r}.

[51] De Soto, *Introductiones Dialecticae*, f. 74^{v}.

But it lacked one important element, namely, a plausible model-theoretic semantics. Modal and intuitionistic logics had been semantically analysed extensively during the 1960s, and so it was natural that attention should be focused on making the same provision for relevant logic. Richard Sylvan (then Richard Routley) seems to have been the first of several authors who hit on the way it could be done. Remember that what is really distinctive of the relevant logics is that they reject EIQ and NAQ, consequences which most blatantly break Belnap's criterion of relevance (see p. 242 above). So it was necessary that the semantics of relevant logic should show that

$$A \,\&\, {\sim}A \nvDash B.$$ [52]

We need, therefore, to find an interpretation under which 'A & ~A' is true and B false. But how can we make 'A & ~A' true, that is, both A and '~A' true together? Is that not impossible?

Kripke had shown in his semantics for the modal logics that nothing is impossible, at least when constructing semantics.[53] All one need do, Sylvan realised, in order to invalidate NAQ, is allow indices (or worlds, or set-ups) in the model which are inconsistent (and consequently others which are incomplete). In other words, we have a set of indices, W, with a total function v from formulae and indices to truth-values so constructed that at some index, a, and for some formula, A,

$$v(A,a) = v({\sim}A,a) = T,$$

i.e., A and '~A' are both true at the index a.

Clearly, if this is to work, we cannot adopt the usual semantical clause

$$v({\sim}A,a) = T \text{ if } v(A,a) = F,$$

i.e., '~A' is true at a if A is false there. What Sylvan did was add an operation $*: W \rightarrow W$ (i.e., a function mapping indices to indices), and require instead that

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[52] We write $X \vDash A$ if A is a logical consequence of X, that is, if A is true in every model of X (i.e., every interpretation which makes X true). $X \nvDash A$ means that A is not a logical consequence of X.

[53] S. Kripke, "Semantical Analysis of Modal Logic, II: Non-normal modal propositional calculi", in *The Theory of Models*, ed. J. Addison, L. Henkin and A. Tarski, Amsterdam: North-Holland, 1965, 206–20.

$$v({\sim}A,a) = T \text{ if } v(A,a^*) = F.$$ [54]

To cut a long story short, careful monitoring of the star-operation showed that the semantics was both coherent and provided a semantics for **R** and **E** (and many other logics such as **B** and **DW**).[55]

It may occur to the reader that there is something disturbingly schizophrenic about this semantics. The reason why a classical logician thinks that EIQ and NAQ are valid is that he thinks the impossibility of true antecedent and false consequent suffice for validity (see p. 234 above). But it is precisely this criterion which is given formal explication in the familiar model-theoretic criterion of logical consequence, that every model of the antecedent should model the consequent. Hence to retain the model-theoretic criterion while at the same time looking to invalidate EIQ and NAQ immediately sets up a tension, a tension which can hardly be said to be removed by allowing the models to make contradictions true and tautologies false. More cogent would be to develop a non-classical model-theoretic account of consequence.

Be that as it may, the ensuing development in the mid-'70s of Sylvan and Meyer's classical relevant logic, **CR**, will help with our medieval hermeneutic puzzle. For Sylvan and Meyer[56] discovered that one can add a new monadic connective '¬' whose truth-condition is the familiar classical Boolean one:

$$v(\neg A,a) = T \text{ if } v(A,a) = F.$$

What was really surprising was that adding '¬', which they called "Boolean negation", to the language of '~' (now called "De Morgan negation"), '&', '∨' and '→' gave a *conservative* extension of relevant logics in which

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[54] See R. and V. Routley, "The semantics of first-degree entailment", *Noûs*, 6, 1972, 335–59, p. 338. The essential idea can be found in J.M. Dunn, *The Algebra of Intensional Logics*, University of Pittsburgh Ph. D., 1966, and in A. Balnicki-Birula and H. Rasiowa, "On the representation of quasi-Boolean algebras", *Bulletin de l'Academie Polonaise des Sciences*, 5, 1957, 259–61.

[55] See, for example, Routley et al., *Relevant Logics and their rivals*, Ch. 4; Read, *Proof Theory and Semantics for Relevant Logic*; *Relevant Logic*, Ch. 5.

[56] See R. Meyer and R. Routley, "Classical Relevant Logics" I and II, *Studia Logica*, 32, 1973, 51–68; 33, 1974, 183–94. The reader will naturally ask here: what does "¬" mean?—does it mean negation? One way to understand the difference between "~" and "¬" is that while "~A" says that A is false, "¬A" actually makes the (one might almost say, metalinguistic) statement that A is not true. See p. 258 below.

$A \,\&\, \neg A \models B$

and

$A \vee B, \neg A \models B.$

What this means is that the valid sequents of **CR** containing only '~', '&', '∨' and '→' are those of **R**; all that the theory of '¬' adds is an epiphenomenal Boolean theory, a surface gloss of classical theory on the relevant base. The new sequents, of course, lack Belnap's variable-sharing property, while it continues to hold in the old vocabulary.

A logic is a formal theory with an interpretation. '~' is the connective designed to formalise negation, and to encapsulate its logical properties. Thus A and '~A' are contradictories. One proof of Gamma consists in showing that we can demand that the real world 0 in any model be consistent, that $0 = 0^*$.[57] But elsewhere in the model, A and '~A' may both be true—for the role of these other worlds or set-ups is to model the antecedent of a consequence (or sequent), to make an assumption for the sake of argument. Supposing that A is true at such an index it does not follow, as de Soto said, that '~A' is false, since both were conceded to be true. That is what is meant by taking an index in the model where both A and '~A' are true. That index, or theory, models our assumption. But there cannot be an index where both A and '¬A' are true. For the truth of '¬A' requires that A be false, and A cannot be both false and true.

What is needed for a valid case of Disjunctive Syllogism is not the *contradictory* of one disjunct of its major premise, but its Boolean negation: having 'A ∨ B', we will be warranted in proceeding to B not by positing the contradictory of A, namely, '~A', but by taking away the truth of A, that is, if we have '¬A'. It is '¬A' which asserts that A is not true. But '¬', we might say, is a technical trick. It is not what we usually express with negation. That is captured in the logics by '~'. Thereby we can explore and reason about various assumptions, made for the sake of argument. As de Soto said, *credo nullum esse impossibile unde sequatur quodlibet.*[58]

57 See J.M. Dunn, "Relevance Logic and Entailment", p. 213; Routley et al., *Relevant Logics and their Rivals*, pp. 387–91. Remember that the semantics is possible-worlds semantics. Logical truth means truth in every model at the real world, 0.

58 D'Ors, "Las Summulae", n. 47, p. 216.

8. Conclusion

I have used certain ideas which have been developed over the last thirty years in relevant logic to try to interpret certain aspects of the medieval debate over the validity of EIQ and NAQ. In classical logic, both Disjunctive Syllogism and Gamma hold; that is, Gamma holds in both thesis and rule form. Only by considering different logics, such as **R** and **E**, can one see that it is possible both to accept the one and reject the other. But do not misunderstand me to be claiming that de Soto, or any other medieval logician accepted **R**, or **E**, or whatever. What they say is almost certainly insufficient to make clear an identification of any precise underlying logic. There are many logics besides **R** and **E** which give good sense to what de Soto says. The important point is that classical logic, **K**, is not one of them.

What I have not tackled is the open question whether a non-classical account of logical consequence, perhaps based on the recurring idea of containment or inclusion of consequent in antecedent, can be developed, inspired by the medieval phrase *de formali intellectu*. But that will have to wait for another occasion.

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